JE

Chern Classes

- Complex vector bundles
- Complex vector bundle structures
- Complex manifolds
- Almost complex structures
- Chern class Axioms
- Chern class constructions
- ex: tautological bundle on $\mathbb{CP}^1$.

of rank n

Def: A complex vector bundle $\zeta : E \to B$ consists of a base B, total space E, and projection map ζ such that:

- each fiber $\zeta^{-1}(b)$ is a complex vector space
- Each point of B has a trivializing neighborhood U and homeomorphism ϕ_U:

$$\phi_U : \zeta^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^n$$

such that

$$\phi_U\big|_{\zeta^{-1}(b)} : \zeta^{-1}(b) \longrightarrow b \times \mathbb{C}^n$$

is complex linear on each fiber

We can perform our usual operations:

- Direct sums
- Tensor products
- Pullbacks.

Complex Structures

As vector spaces:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "\text{underlying real structure}"] & \mathbb{R}^{2n} \arrow[l, bend left, "\text{define "multiplication by } i\text{"}"] \end{tikzcd}$$

We want a real linear map $J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 \overset{?}{=} -Id.$

Then $\mathbb{R}^{2n}$ is a complex vector space, under
$$(a+bi)\cdot v = av + bJ(v).$$

Similarly, for vector bundles:

$$\begin{tikzcd} \zeta^n \arrow[r, bend left, "\text{underlying real structure}"] & \zeta_{\mathbb{R}}^{2n} \arrow[l, bend left, "\text{define a complex vector bundle structure}"] \end{tikzcd}$$

complex rank n real rank $2n$

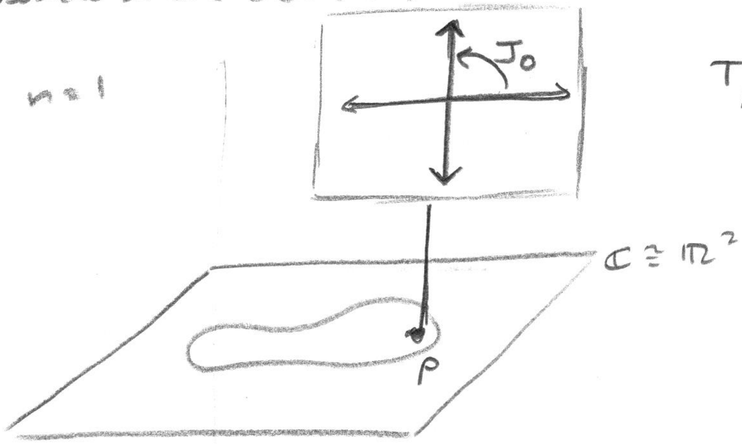
<u>Def</u>: A <u>complex vector bundle structure</u> J on $\zeta^{2n}: E \to B$, a rank $2n$ real vector bundle, is a continuous map $J: E \to E$ such that:

- $J|_{\zeta^{-1}(b)}$ is a real linear map from $\zeta^{-1}(b)$ to itself
- $J^2(v) = -v$ for each vector in the bundle

Complex Manifolds

For any open set $U \subseteq \mathbb{C}^n$, the tangent bundle TU is $U \times \mathbb{C}^n$. It has a canonical complex vector bundle structure $J_0(u,v) = (u, iv)$.

ex: $n=1$



Given open subsets $U \subseteq \mathbb{C}^n$, $U' \subseteq \mathbb{C}^p$ and a (real) smooth map

$$f : U \longrightarrow U',$$

we can check whether f is holomorphic (with respect to the (canonical complex structures on U and U') by checking whether its differential is complex linear: $df \circ (J_0)_U = (J_0)_{U'} \circ df$

This is equivalent to checking that each coordinate of $f = (f_1, \dots, f_p)$ satisfies the Cauchy-Riemann equations w.r.t. each complex coordinate $z_1, \dots, z_n$.

Def: A <u>complex manifold</u> M^n is a topological space M with an atlas of complex charts:

$$\{\Phi_\alpha : \underset{\mathbb{C}^n}{\overset{\cap\!|}{U_\alpha}} \xrightarrow[\sim]{\text{homeo}} \underset{M}{\overset{\cap\!|}{V_\alpha}}\}$$

such that:

- the V_α cover M
- the change of chart morphisms

$$\Phi_\beta^{-1}\Phi_\alpha : \underset{\mathbb{C}^n}{\overset{\cap\!|}{\Phi_\alpha^{-1}(V_\alpha \cap V_\beta)}} \longrightarrow \underset{\mathbb{C}^n}{\overset{\cap\!|}{\Phi_\beta^{-1}(V_\alpha \cap V_\beta)}}$$

are holomorphic.

Equivalently, we can start with a real smooth $2n$-fold M with a complex vector bundle structure J on its tangent bundle — this is called an <u>almost complex structure</u> on M. M is a <u>complex manifold</u> if, for each $x \in M$, there exists a neighborhood V of x, an open set $U \subseteq \mathbb{C}^n$, and a diffeomorphism $\Phi : U \to V$ such that $d\Phi \circ J_0 = J \circ d\Phi$ (i.e., the almost complex structure in a neighborhood of x is induced by Φ from the complex canonical structure on U).

The Newlander-Nirenberg theorem gives conditions for when an almost complex structure on a real $2n$-fold M arises from M actually being a complex manifold. It is a very hard theorem

Chern Classes:

Chern classes are characteristic classes on complex vector bundles, analogous to Steifel-Whitney classes

Axiomatically: Given a complex vector bundle $\zeta : E \to B$,

(i) $c_i(\zeta) \in H^{2i}(B, \mathbb{Z})$

(ii) Naturality: $c_i(f^*\zeta) = f^*(c_i(\zeta))$

(iii) Whitney Sum: $c(\zeta \oplus \eta) = c(\zeta) \cdot c(\eta)$

(iv) Nontriviality: $c(\gamma^1) = 1 - a$ where a is the Poincare dual of $[\mathbb{C}P^{k-1}]$ in $\mathbb{C}P^k$.

How do we construct?

- We will construct recursively using Euler classes.

Fact: Every complex bundle has orientable underlying real bundle.

Pf. Start with a complex basis $a_1, \dots, a_n$. On the real basis $a_1, ia_1, a_2, ia_2, \dots, a_n, ia_n$, the complex elementary row operations have the following effects:

(i) Scaling $a_j \mapsto (b+ci)a_j$. On the real basis, this sends

$$a_j \longmapsto ba_j + c(ia_j)$$
$$ia_j \longmapsto -ca_j + b(ia_j),$$

which changes the determinant by $b^2-(-c^2) = b^2+c^2 > 0$.

(ii) Adding one coordinate to another - sending

$$a_j \mapsto a_j + a_k$$

just sends:

$$a_j \longmapsto a_j + a_k$$
$$ia_j \longmapsto ia_j + ia_k$$

, which doesn't change determinant.

(iii) Permuting coordinates - swapping a_j and a_k swaps both $a_j \leftrightarrow a_k$ and $ia_j \leftrightarrow ia_k$ in the real basis, which doesn't change sign.

So any change of complex basis preserves the sign of determinant in the real basis.

Alternately, $GL_n(\mathbb{C})$ is connected, giving a path from any real basis which arises from a complex basis to any other such basis.

Def: On a rank n bundle ζ, the Chern classes $c_i(\zeta)$ are:

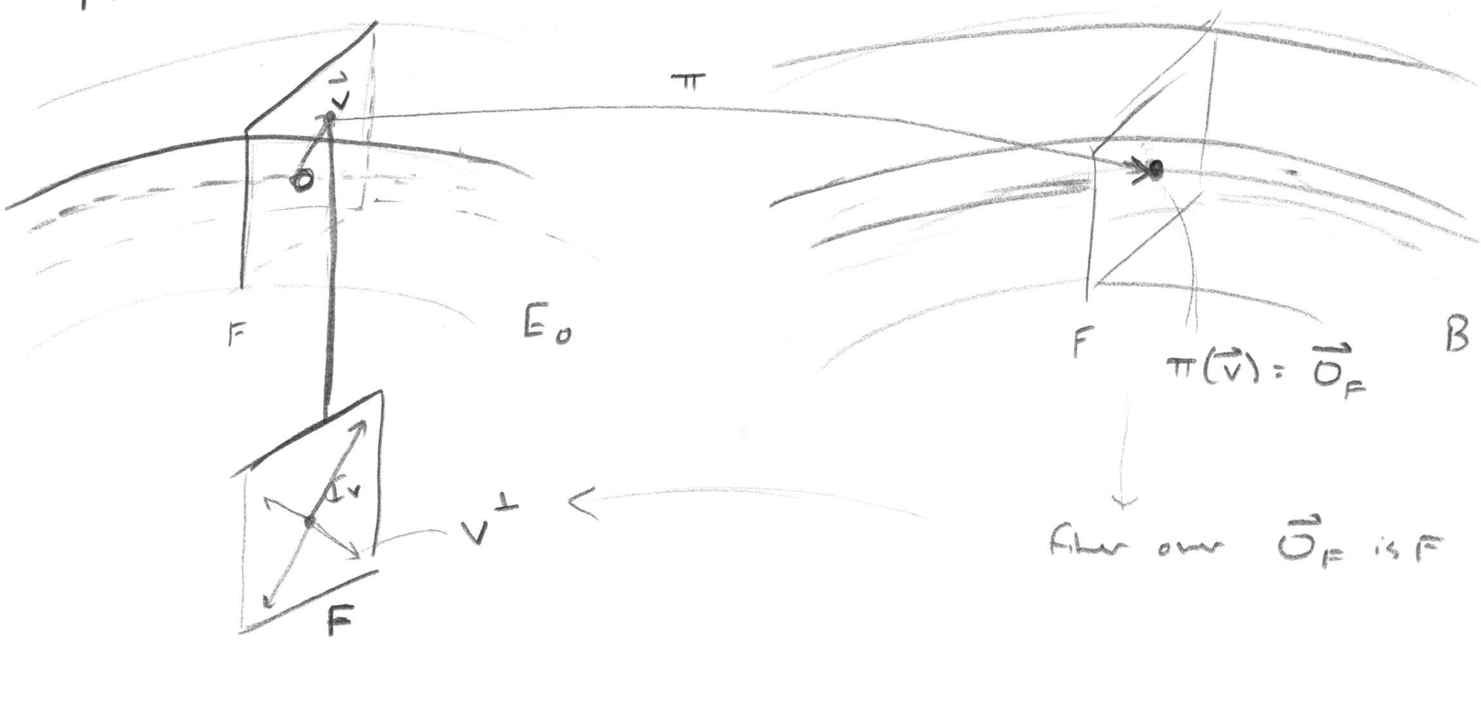
- $c_i(\zeta) = 0$, $i > n$
- $c_n(\zeta) = eu(\zeta_{\mathbb{R}})$, the Euler class of the underlying real bundle
- $c_i(\zeta) = (\pi^*)^{-1} c_i(v^\perp)$, where $v^\perp$ is a rank $n-1$ bundle on $E_0 = \zeta - B$, and $\pi : E_0 \to B$ is projection.

How is $v^\perp$ constructed?

We can pull back the bundle ζ using the map π:

$$\begin{tikzcd} \pi^*\zeta \arrow[r] \arrow[d] & \zeta \arrow[d] \\ E_0 \arrow[r, "\pi"] & B \end{tikzcd}$$

A point in E_0 is specified by a fiber F of ζ, with a non-zero vector $\vec{v}$ in that fiber. When we pull back, the fiber over $(F, \vec{v})$ is another copy of F.



Then this fiber over (F, v) decomposes as a direct sum of $\mathbb{C}\vec{v} \oplus v^{\perp}$ (using a Hermitian metric on ζ).

This gives us a decomposition

$$\underset{\text{rank } n}{\pi^*\zeta} = \underset{\text{rank } 1}{\operatorname{span}(v)} \oplus \underset{\text{rank } n-1}{v^{\perp}},$$

One can also define $v^{\perp} = \pi^*\zeta / \operatorname{span}(v)$.

We have Chern classes for $v^{\perp}$, by recursion.

But cohomology classes pull back: we only have a map

$$\pi^* : H^*(B) \longrightarrow H^*(E_0)$$

However, for $i < 2n-1$, this is an isomorphism: the Gysin exact sequence associated to the real oriented bundle $\zeta_{\mathbb{R}}$ of rank $2n$ is

$$\cdots \longrightarrow H^{i-2n}(B) \xrightarrow{\cdot e} H^i(B) \xrightarrow{\pi^*} H^i(E_0) \longrightarrow H^{i-2n+1}(B) \longrightarrow \cdots$$

Since $H^{i-2n}(B)$ and $H^{i-2n+1}(B)$ are 0 for $i < 2n-1$, we have that for $0 \leq i \leq 2n-2$ the LES looks like

$$\cdots \to 0 \longrightarrow H^i(B) \xrightarrow{\pi^*} H^i(E_0) \longrightarrow 0 \longrightarrow \cdots;$$

so that $\pi^* : H^i(B) \xrightarrow{\sim} H^i(E_0)$ for $i \leq 2n-2$.

Thus it makes sense to talk about

$$(\pi^*)^{-1}(c_i(v^{\perp})).$$

Rmk: This means that on a complex line bundle γ,

$c(\gamma) = 1 + eu(\gamma_{\mathbb{R}})$.

ex: Tautological line bundle γ^1 on $\mathbb{CP}^1$.

$c(\gamma^1) = 1 + eu(\gamma_{\mathbb{R}})$

Recall from last week that we can compute the Euler class of a bundle by taking the Poincaré dual of the fundamental class of the zero locus of a generic section.

Apply this to the **dual bundle** $(\gamma^1)^*$: the fiber over the point $\ell \in \mathbb{CP}^1$ is $\mathrm{Hom}(\ell, \mathbb{C})$.

Taking a linear functional $\Phi: \mathbb{C}^2 \to \mathbb{C}$, the restriction $\Phi|_\ell$ is in $\mathrm{Hom}(\ell, \mathbb{C})$. So the map

$s: \ell \mapsto \Phi|_\ell$ gives us a section of $(\gamma^1)^*$.

It's only zero at the line $\ker(\Phi)$, (which is a line by rank-nullity).

So $Z = s \cap \mathbb{CP}^1 = \{pt\}$, and $PD[Z] = PD[pt] = PD[\mathbb{CP}^0] = a$.

Then $eu((\gamma^1)^*_{\mathbb{R}}) = a$

But $(\gamma^1)^*$ is the "conjugate bundle" to γ^1 - taking a Hermitian inner product on γ^1, the map

$v \longmapsto \langle -, v \rangle$ from γ^1 to $(\gamma^1)^*$

is a conjugate complex linear map. This means that $\gamma^1_{\mathbb{R}}$ has real basis a_1, ia_1 in each fiber, which gets mapped to the real basis $\langle -, a_1 \rangle, -i\langle -, a_1 \rangle$ in each fiber of $(\gamma^1)^*_{\mathbb{R}}$.

This is orientation reversing, since the canonical orientation on $(\gamma_1^{1})^*_{\mathbb{R}}$ is $\langle -, a_1 \rangle$, $i\langle -, a_1 \rangle$.

So $eu(\gamma^1_{\mathbb{R}}) = -eu((\gamma^1)^*_{\mathbb{R}}) = -a$